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The intensity of classical bright solitons propagating in linearly coupled identical fibers can be distributed either in a stable symmetric state at strong coupling or in a stable asymmetric state if the coupling is small enough. In the first case, if the initial state is not the equilibrium state, the intensity may switch periodically from fiber to fiber, while in the second case the a-symmetrical state remains forever, with most of its energy in either fiber. The latter situation makes a state of propagation with two exactly reciprocal realizations. In the quantum case, such a situation does not exist as an eigenstate because of the quantum tunneling between the two fibers. Such a tunneling is a purely quantum phenomenon which does not exist in the classical theory. We estimate the rate of tunneling by quantizing a simplified dynamics derived from the original Lagrangian equations with test functions. This tunneling could be within reach of the experiments, particularly if the quantum coherence of the soliton can be maintained over a sufficient amount of time.

Lead Paragraph

Usually solitons in optical fibers are assumed to be classical (= non quantum) objects because they are made of a large number of photons. Nevertheless there exist quantum effects without classical counterpart, like the tunneling under a potential barrier. We investigate one possible realization of such a quantum tunneling with solitons as basic entities. Specifically, we consider a soliton propagating in two linearly coupled fibers that are assumed identical. It has been known for some time that, at small enough coupling, asymmetric solitons only can propagate and be stable. The amplitude of such asymmetric solitons is predominantly in either fiber and remains there forever classically. This makes, for a given energy, two possible steady states exactly symmetrical with respect to each other under permutation of the two fibers. In the quantum version of the same problem, the two solitons merge into a single quantum state sharing a quantum amplitude spread between the two fibers, because of the possibility of *quantum* tunneling from one fiber to the other. We study this problem thanks to a reduced set of equations derived from the full set of coupled nonlinear PDE's by choosing convenient trial functions for the classical soliton dynamics. Thanks to this choice, the bifurcation pattern of the soliton solution in the coupled fibers is well reproduced. Because the trial set generates dynamical equations with a Lagrange structure, this Lagrangian system is relatively easy to quantize. To obtain the quantum amplitude of transmission by tunneling under the barrier, one replaces the original Hamiltonian system by its Euclidean counterpart. Orders of magnitude relevant for a possible physical application are given.

PACS numbers:

I. INTRODUCTION

Generally speaking a soliton is a localized solution of an equation for the propagation of a field envelope. It stays localized under the opposite effects of linear dispersion tending to spread the wave and of nonlinearity making the wave steeper. We make one more step by considering this soliton as a 'true' particle, that is by seeing it as a classical object that should be ultimately quantized to keep the consistency of our view of the physical world. This is of course not a new idea, see for instance the review [1] on the quantization of various nonlinear equations for classical fields. Quantum effects are irrelevant for macroscopic phenomena like solitary waves in a water channel. However, there is an instance of solitonic physics where quantization could bring significant new effects, namely the propagation of optical solitons in fibers: there the amplitude of the wave may be small enough to yield solitons with not too large action, measured in units of Planck's constant $\frac{h}{2\pi}$. If the action is much larger than this quantum unit one is in the classical regime, many trajectories contribute to the saddle point of the Feynman integral [2] and quantum interferences between coherent quantum states become practically impossible. Conversely, if the action is not too large compared to this quantum unit, one could observe quantum phenomena as tunneling and interferences. Moreover,

even if the quantum state has a coherence time shorter than the tunneling time, there is still quantum tunneling, but at a reduced rate because the build-up of the state on the other side of the barrier is slowed down [3]. Below we show that quantum tunneling of a soliton may occur between two weakly coupled fibers, and we discuss the possibility of quantum interference between the two states carried by each fiber. The starting point of our study is the well established result that *classically* a soliton injected in a given fiber cannot switch to the other fiber, when the coupling is less than a certain critical value. In this range the soliton evolves towards the stable a-symmetric solution having its energy predominantly in the initial fiber [4]. There exist another stable solution which is obtained by permutation of the two fibers. At small coupling the two a-symmetrical states are separated by a finite barrier that cannot be crossed classically. We predict that this may be wrong in practice because of quantum tunneling. Our derivation is based on the calculation of the tunneling probability which writes in the WKB approximation, as $T = \exp(-2S/\hbar)$, where S is the physical action associated to tunneling [5]. The quantity S/\hbar may be small, even for a pulse with a large number of photons, because all dynamical phenomena we consider, like the balance between the nonlinearity, the group velocity dispersion and the coupling, imply small perturbations to the dominant effect resulting from the linear dispersionless terms of Maxwell's equations. The perturbations we consider are as small as the nonlinear term $n_2 I$ with respect to the dominant term n_0 , in the expansion of the refractive index $n = n_0 + n_2 I$ for the Kerr medium of the fiber, I being the optical intensity. In this sense a soliton is a bound state of photons: photons are attracted to each other by the focusing nonlinearity. The soliton resembles the atom of a heavy element which is a quantum object made of many electrons and nucleons. Atomic physics has also to do with energies much smaller than the rest energy of the particles (electrons and nucleons) making the atom. Another idea of atomic (and quantum!) physics is relevant for our goal: If the atom remains in its ground state, it is able to make interferences with a wavenumber depending on its mass and velocity only, independently of the details of the state of its electrons and nucleons. In our study we assume the absorption and change of frequency of the photons by inelastic and/or Raman scattering to be negligible, and discuss the role of these effects in interference experiments in the last section.

Let us outline the organization of this paper. In the Section II we introduce the classical model of propagation of solitons in fibers. First we deal with the single fiber, then with the two coupled fibers. There is nothing new here and we focus on what is relevant for us, namely the bifurcation diagram as a function of the coupling. For strong coupling the stable solution is symmetric (with intensity equally shared between the two fibers). Below a critical coupling, the classical prediction is that the stable soliton is a-symmetric, with a high amplitude in one fiber and a small amplitude in the other, as said above. The details of the transition are a bit complex because it is subcritical. This has been studied [4] by direct numerical solution of the coupled PDE's describing this problem (equations (6) and (7) below). However interesting it is, this model presents some difficulties for our quantization problem. Therefore, in the next section III we outline another approach to the same problem, namely we use the Lagrange formalism to compute approximate solutions with trial functions (instead of the full unknown solution). Thanks to an appropriate choice of these functions, the pattern of bifurcations of the asymmetric to symmetric solitons, known from the direct numerical simulations, is well recovered. We use the same trial functions as Malomed et al. [6] who discuss very thoroughly the general issue in a paper that we recommend to the interested reader.

As explained in section IV, this 'trial dynamics' is used then to quantize the system. This is of course not exact, but requires far less formalism than the full quantization of the two coupled nonlinear field equations. Thanks to this method, one can use standard results and methods of quantum mechanics for systems with a few degrees of freedom (as opposed to field theories). In particular, we can compute the trajectory under the potential barrier, found by multiplying the propagation variable z , and the Hamilton-Jacobi action S by i . This yields a well defined problem of Hamiltonian mechanics, called sometimes the Euclidean version of the initial problem. The tunneling factor is derived from the action of the heteroclinic trajectory joining the two equilibria: a stable equilibrium in the original Hamiltonian system remains an equilibrium in the Euclidean one, but it becomes unstable there. It turns out that the tunneling probability depends algebraically on the coupling between the two fibers. This is a significant remark for possible applications, because it yields a much smoother dependence with respect to the coupling than the usual exponentially small tunneling amplitudes.

The last section summarizes the main results of this paper, discusses the possibility of interferences and presents some ideas for possible applications. Quantitative predictions rest on the rather complex problem of turning back from the dimensionless equations used throughout this work to quantities with a physical dimension, something done in the Appendix.

The mathematical model for the dissipationless propagation of optical solitons in one fiber is the *classical* (= non quantum) nonlinear Schrödinger equation:

$$i \frac{\partial E}{\partial z} + \alpha \frac{\partial^2 E}{\partial t^2} + \beta |E|^2 E = 0. \quad (1)$$

Even though this equation is called nonlinear Schrödinger (NLS), it does *not* mean at all that it makes a quantum system. It only resembles the usual Schrödinger equation, but it describes a purely classical field, exactly as Maxwell's equations do for an EM field. This equation is written with real coefficients α and β carrying a physical dimension to make possible the discussion (see Appendix) of the order of magnitude of the physical effects to be expected. The field E is the complex amplitude of the electric field in the wave. We take it as a scalar, although polarization effects could be brought into the picture in principle. This equation is derived in the Fresnel approximation, assuming that the changes of amplitude along the fiber are much slower and on much longer scales than the oscillations of the optical field itself and it is also written in the frame of reference moving with the speed of the envelop of the wave, where the position variable is z . For α and β real, this equation has a Lagrange-like structure. It cancels the first order variation of the 'action'

$$\mathcal{S} = \int dz \int dt \left[\frac{i}{2} \left(\bar{E} \frac{\partial E}{\partial z} - E \frac{\partial \bar{E}}{\partial z} \right) - \alpha \left| \frac{\partial E}{\partial t} \right|^2 + \frac{\beta}{2} |E|^4 \right]. \quad (2)$$

In this equation \bar{E} is the complex conjugate of E . The writing of the action in equation (2) brings in an important problem, because it is not 'the' physical action. Such a physical action has to have the dimension of the product of an energy and of a time. Therefore, the action written in equation (2) cannot be an action from the point of view of physical dimensions. The physical action of the EM field is proportional to \mathcal{S} , its derivation is postponed to the Appendix. An overall constant multiplying factor does not change the Euler-Lagrange equations, but it is crucial when quantizing this system because this relies on a comparison between the action and \hbar , two quantities with the same physical dimension.

By rescaling $E \rightarrow u\beta^{-1/2}$ (assuming β positive to be in the focusing case where solitons exist), and $t \rightarrow t(2\alpha)^{1/2}$, one obtains the dimensionless nonlinear Schrödinger equation:

$$i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial t^2} + |u|^2 u = 0. \quad (3)$$

This equation has a number of interesting symmetries. In addition to the Galilean invariance (if $u(z, t)$ is a solution, then $u(z, t - z/C)e^{\frac{i}{2}(t - \frac{z}{C})}$ is also a solution), it has a dilation symmetry: if $u(z, t)$ is a solution and μ an arbitrary real number, then $\mu u(z\mu^2, t\mu)$ is also a solution. It has a two parameters family of soliton solutions:

$$u_0 = \frac{\nu e^{i\varphi}}{\cosh(\nu t)} = \nu e^{i\varphi} \text{sech}(\nu t). \quad (4)$$

In this solution, ν is any real number and the phase φ is $\varphi = \frac{1}{2}\nu^2 z + \varphi_0$, with φ_0 arbitrary constant phase.

Note that z plays here the same role as the time in the usual Schrödinger equation. Among the conserved quantities associated to any solution of the NLS equation, let us write the "energy"

$$\mathcal{H} = \frac{1}{2} \int dt \left[\left| \frac{\partial u}{\partial t} \right|^2 - |u|^4 \right]. \quad (5)$$

Suppose now that, instead of a single optical fiber, we have two identical coupled fibers, and that the coupling is linear and preserves the symmetry between the fibers. The propagation of solitons in this system has been studied in the last fifteen years [4], [6]-[10].

To describe the two coupled fibers supporting solitons we introduce two focusing NLS equations, written in a dimensionless form:

$$i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial t^2} + |u|^2 u = -\kappa v, \quad (6)$$

and

$$i \frac{\partial v}{\partial z} + \frac{1}{2} \frac{\partial^2 v}{\partial t^2} + |v|^2 v = -\kappa u, \quad (7)$$

where κ is the strength of the linear coupling, and we define the "mass"

$$Q = \int (|u|^2 + |v|^2) dt. \quad (8)$$

which is a constant of motion. Consider solutions of the form $u(z, t) = U(t)e^{iqz}$ and $v(z, t) = V(t)e^{iqz}$. Because of the common phase factor e^{iqz} the z -dependence cancels out and the two functions $U(t)$ and $V(t)$ are solutions of the two coupled ordinary differential equations:

$$\begin{cases} -qU + \frac{1}{2} \frac{d^2 U}{dt^2} + |U|^2 U = -\kappa V \\ -qV + \frac{1}{2} \frac{d^2 V}{dt^2} + |V|^2 V = -\kappa U. \end{cases} \quad (9)$$

For the solution to decrease to zero when t tends to plus or minus infinity one must have $q > 0$. Furthermore the sign of κ can be changed by changing U into $-U$ for instance and keeping V the same. We choose κ positive that corresponds to in-phase stationary solutions (U, V) , the out of phase ones being unstable [4].

This set of equations has been studied numerically and analytically [4]. An exact calculation shows that the symmetric solution $U = V$ always exists and is linearly stable in the range $\frac{Q}{\sqrt{\kappa}} \leq \frac{8}{\sqrt{3}}$. For higher values of this ratio, the symmetric solution loses its stability and an a-symmetric solution branches off. While it is not explicitly mentioned in [4], the subcritical character of the bifurcation can be deduced from Fig.11 of the paper by Akhmediev and Soto-Crespo(1994) when using $\frac{Q}{\sqrt{\kappa}}$ as control parameter and $\frac{q}{\kappa}$ as order parameter, i.e. by rotating the figure.

Consequently no stable and weakly asymmetric solutions branch off the unstable symmetric soliton for κ slightly smaller than the onset of linear stability, although an unstable asymmetric solution branches off at values of κ slightly larger than the critical one. Furthermore a branch of stable asymmetric solitons goes continuously from $\kappa = 0$ to a finite coupling, slightly larger than the value of linear instability of the symmetric soliton. The stable asymmetric soliton disappears by a saddle-node bifurcation for a value of the coupling that is, by a numerical coincidence, very close to but smaller than the onset of linear stability of the symmetric soliton. At this saddle-node bifurcation the unstable and stable a-symmetric solutions merge to disappear at smaller values of $\frac{Q}{\sqrt{\kappa}}$.

In the numerical investigations of this problem an interesting phenomenon comes into play, namely the radiation of energy at large distances of the solitons. The amount of radiation is stronger when the initial conditions are further away from a stable solution [4]. This radiation happens in the far wings of the time dependent amplitude profiles $(|u(t)|, |v(t)|)$, where the full equation reduces to its linear part. Although very strongly dispersive this describes radiation by wave packets of ever increasing width, but carrying nevertheless energy and eventually other invariants to infinity. Such a coupling between a localized system and the infinitely many degrees of freedom of a radiating field may lead to irreversible effects [11]. It shows how subtle may be the distinction between 'dissipative' and 'nondissipative' systems as soon as one goes beyond the obvious. Irreversible process due to radiation may not even require an infinitely extended physical space. They may also take place in the reciprocal (or momentum) space by cascade of energy toward smaller and smaller scales, a typically nonlinear phenomenon [12]. We plan to come back to the issue of the effect of radiation on quantum phenomena in the present problem. We shall neglect this kind of effect in the following, since they cannot be taken into account within our simple formalism. Even though the radiative losses are present, it was shown by Fadeev and Korepin [1] that they do not destroy the solitons in a single fiber, when they are included in the quantized version of the NLS equation.

In the coming section we shall derive a reduced set of equations describing the propagation of soliton in coupled fibers. Indeed this reduction from the original PDE's to a set of coupled ODE's cannot be quantitatively exact. However with the same choice of trial functions as Malomed et al.[6] we obtain at least a reduced system with the right pattern of bifurcation at decreasing coupling. The fundamental interest of this reduction is that it allows us to quantize the dynamical system rather straightforwardly.

III. CLASSICAL PROPAGATION OF SOLITONS IN COUPLED FIBERS: THE REDUCED DYNAMICS

Because of the lack of analytical solution in general, we follow an idea used already by various authors, that allows to understand in a fairly detailed way the results of the direct numerical simulation by using an analytical approach. This follows the general method of research of extrema of functionals by trial functions: dynamics can be reduced to a minimization problem, then one restricts the function space where this minimization is done to a space of trial functions depending explicitly on a few parameters and one studies the dynamical properties in this reduced space. Since we know the results of the direct numerical simulations it is in principle possible to check the quality of the approximation by comparing its predictions and the 'exact' results. This is necessary because the method of trial

functions does not rely on a small or large parameter and so cannot hope to be ‘exact’ or close to exact in the usual mathematical meaning of the word. The papers by Malomed et al. [6] discuss in depth the choice of the trial functions. We shall not reproduce this discussion here where we take their set of ‘optimized’ trial functions, following as much as possible their notations.

The starting point is the writing of the action for the coupled NLS equations:

$$\mathcal{S}_{NLS} = \int_{z_1}^{z_2} dz \int dt \left[\frac{i}{2} \left(\bar{u} \frac{\partial u}{\partial z} - u \frac{\partial \bar{u}}{\partial z} \right) + \frac{i}{2} \left(\bar{v} \frac{\partial v}{\partial z} - v \frac{\partial \bar{v}}{\partial z} \right) - \frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 + \frac{1}{2} |u|^4 - \frac{1}{2} \left| \frac{\partial v}{\partial t} \right|^2 + \frac{1}{2} |v|^4 + \kappa (u \bar{v} + v \bar{u}) \right]. \quad (10)$$

As it can be checked the action \mathcal{S}_{NLS} is proportional to $(z_2 - z_1)$ whenever the functions $u(z, t)$ and $v(z, t)$ are stationary solutions (with respect to the variable z) of the two coupled NLS equations (6) and (7) or functions proportional to the same phase factor e^{iqz} . The problem we consider now is how does the coupling change the propagation of solitons. For that purpose we reduce the dependence with respect to t to an imposed form (the trial function) with arbitrary z -dependent coefficients, the trial parameters. Putting this trial form into the action integral and performing the integration over the variable t yields a functional of the parameters of the trial function that are themselves functions of z . Doing now the variation with respect to those functions, one finds at the end a set of ODE’s for functions of z only.

The choice of the trial functions is inspired by the soliton solution in a single fiber and it respects the symmetry between the two fibers. Following Uzunov et al.[6] one takes:

$$u(z, t) = a(z) \sqrt{\eta(z)} \operatorname{sech}[\eta(z)t] \cos(\Theta(z)) \exp[i(\Phi(z) + \Psi(z) + q(z)t^2)], \quad (11)$$

and

$$v(z, t) = a(z) \sqrt{\eta(z)} \operatorname{sech}[\eta(z)t] \sin(\Theta(z)) \exp[i(\Phi(z) - \Psi(z) + q(z)t^2)]. \quad (12)$$

In the case of a single fiber carrying a soliton of amplitude u , the trial function $u(z, t)$ becomes the exact one-soliton solution with $a = \sqrt{\eta}$, $\Theta = q = 0$, $\Phi = z\eta^2/2$ and Ψ constant. It is important to notice here that the angle Θ is for describing the balance between the two fibers, although the angles Φ and Ψ have a physical meaning independent on the trial function, being related to the phase of the functions u and v . The angle Θ could be replaced by another parameter in another trial function, not necessarily a circular function. Inserting this trial form into the action \mathcal{S}_{NLS} and performing the integration over t , which is possible because the dependence with respect to t is fully explicit in the trial functions, one finds a reduced action that is itself the integral over z of the Lagrange function:

$$\mathcal{L} = 2\kappa a^2(z) \cos(2\Psi) \sin(2\Theta) - 2a^2 \cos(2\Theta) \frac{d\Psi}{dz} - \frac{1}{3} a^4 \eta \sin^2(2\Theta) + \frac{2}{3} a^4 \eta - \frac{1}{3} a^2 \eta^2 - 2a^2 \frac{d\Phi}{dz} - \frac{a^2 \pi^2}{6\eta^2} \left(\frac{dq}{dz} + 2q^2 \right). \quad (13)$$

Up to obvious change in notations (from our z to ζ , from κ to K , etc.) this Lagrange function is identical to the one written by Uzunov et al. [6] but for a misprint in their paper where the term q^2 in the last parentheses became q^4 without harming the rest of their calculation. The parameters of the trial function are five functions of z : a , Θ , Ψ , η and q . The equations of motion for those five functions are derived by variation of the action, namely the integral over z of \mathcal{L} . They read:

$$\frac{da^2}{dz} = 0, \quad (14)$$

derived by variation with respect to Φ , and

$$\begin{cases} \frac{d\Theta}{dz} = -2\kappa \sin(2\Psi) \\ \sin(2\Theta) \frac{d\Psi}{dz} = \frac{a^2}{3} \eta \sin(2\Theta) \cos(2\Theta) - \kappa \cos(2\Psi) \cos(2\Theta) \\ \frac{d\eta}{dz} = -2q\eta \\ \frac{dq}{dz} = -2q^2 + \frac{2}{\pi^2} [\eta^4 - a^2 \eta^3 (1 - \frac{1}{2} \sin^2(2\Theta))] \end{cases} \quad (15)$$

derived by variation with respect to Ψ , Θ , q , and η respectively

The parameter a^2 can be absorbed in the redefinition of κ and will be set to 1 below (that corresponds to a mass $Q = 2$).

The soliton solutions are z -independent solutions of this set of equations. There are two classes of soliton solutions in this model, depending on the coupling parameter. For any coupling there exists a symmetric soliton, with equal intensity in both fibers, i.e. $\Theta = \frac{\pi}{4}$. At small coupling this symmetric solution is unstable against asymmetric soliton.

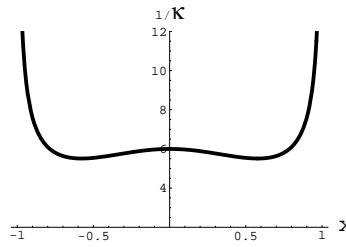


FIG. 1: Inverse of coupling coefficient κ^{-1} versus the stationary values of $x = \cos(2\Theta)$. For large coupling, i.e. $\kappa > 1/6$, the symmetric solution, $x = 0$, is the only one stable.

Such an asymmetric soliton is found by canceling the z -derivatives in equations (15) and choosing $\cos(2\Psi) = 1$. This yields the relation between the coupling coefficient κ and the balance parameter for the intensity $x = \cos 2\Theta$

$$\kappa^{-1} = \frac{6}{(1+x^2)\sqrt{1-x^2}}, \quad (16)$$

which is illustrated in Fig.1. Using the trial functions, the bifurcation in the set of possible solutions is found to occur at the critical value, $\kappa_c = 1/6$ which is 11 per cent less than the exact value [4], $\kappa_c = \sqrt{3}/4$. Moreover the bifurcation is slightly subcritical, in good agreement with the NLS results [6]. Close to the bifurcation point there is a small range of values of the coupling, $\frac{1}{6} \leq \kappa \leq \frac{4}{9\sqrt{6}}$ ($0.167 \leq \kappa \leq 0.181$) where there are three sets of solutions: the symmetric solution that is linearly stable, and two pairs of asymmetric solutions, one linearly stable and another linearly unstable. The branch of stable asymmetric solution does not merge smoothly with the symmetric solution, but disappear when it has still a finite amplitude. The main conclusion that we shall draw here is that this set of trial functions reproduces well the pattern of bifurcation of the exact model. This makes it a good candidate for studying the quantum tunneling.

Before to start this study, let us explain how we managed to define a quantity related to the usual potential energy of a mechanical system. Although this is not strictly necessary it helps to draw various quantities relevant for analyzing the tunneling by making a connection, however loose it is, with the familiar notions of barrier and of barrier crossing.

The ‘potential energy’ is derived from the total energy associated to the dynamical system under consideration, namely the equations (14) to (15). An expression for this energy is given by Uzunov et al. With $a^2 = 1$ it becomes:

$$\mathcal{H}_{\text{trial}}^{\text{cl}} = -2\kappa \cos(2\Psi) \sin(2\Theta) + \frac{1}{3}\eta \sin^2(2\Theta) - \frac{2\eta}{3} + \frac{\eta^2}{3} + \frac{\pi^2 q^2}{3\eta^2}. \quad (17)$$

Note that the Lagrangian (13) includes terms linear with respect to first derivatives (with respect to z). It means that the two successive operations of choosing trial functions and averaging over the retarded time t , lead from the Lagrangian formalism to the Hamiltonian one, with

$$\mathcal{S} = \int \left(\sum_{i=1,2} p_i dq_i - \mathcal{H} dz \right) \quad (18)$$

The first term in the r.h.s. of equation (18) will be the one responsible for the Euclidian action derived in the next section. As already noticed by Uzunov et al., equations (14) to (15) are the Hamilton equations of a two-degrees of freedom system. The two pairs of conjugate variables are $\{\Psi, 2x = 2\cos(2\Theta)\}$, and $\{q, y = \frac{\pi^2}{6\eta^2}\}$, i.e. the phase and amplitude differences, as well as the chirp and width, respectively. We are interested in the value of \mathcal{H} for steady states, that turns out to be a simple function of the coupling κ . That should give an idea of how the energy changes when the variables are different of their values in the steady state(s). In order to preserve the connection with a potential energy in the usual sense we impose that, at the equilibrium points, this ‘potential’ energy is at an extremum. This is realized (probably not uniquely) by plugging into \mathcal{H} the values of q and Ψ at the various equilibria, that is $\Psi = 0$ and $q = 0$ to cancel the conjugate momenta. This yields:

$$\mathcal{H}_{\text{pot}} = -2\kappa \sin(2\Theta) + \frac{1}{3}\eta \sin^2(2\Theta) - \frac{2\eta}{3} + \frac{\eta^2}{3}. \quad (19)$$

This ‘potential’ energy depends on two parameters, $x = \cos(2\Theta)$ and η , and it is plotted in Fig. 2 for various coupling strength to show the bifurcation of the equilibria from a single equilibrium at large coupling, Fig. 2(a), to

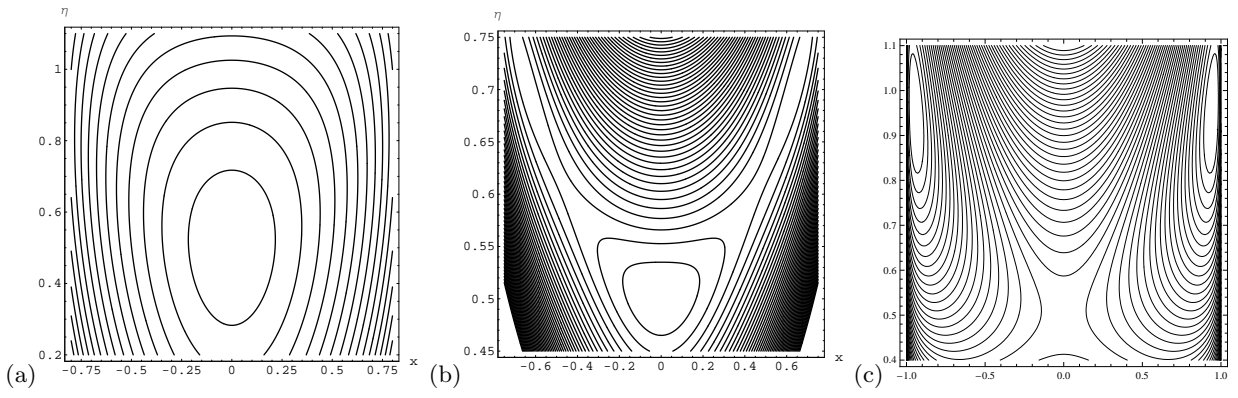


FIG. 2: Level lines of the potential in the plane (x, η) , (a) for $\kappa = 0.5$, only the symmetric solution $x = 0$ is stable, (b) for $\kappa = 0.18$ which belongs to the subcritical domain in Fig.1 where symmetric and a-symmetric solutions are stable, (c) for $\kappa = 0.1$, only two asymmetric solutions are stable

a more complex pattern, as the coupling decreases. In particular, some sort of barrier is evident in Fig. 2(c). It separates the two deep minima of the potential lying each in the vicinity of $(\pm 1, 1)$. Each minimum corresponds to one of the stable asymmetric soliton, although the unstable symmetric soliton at $(0, 1/2)$ is a saddle point of the potential energy.

The above picture illustrates the known results: classically there is no way for a soliton initially in a given fiber, to escape through the other fiber, at low coupling, because of the barrier. Before to present the quantum version of this problem, let us precise what is the low coupling range in terms of physical quantities. Note first that the low-coupling range writes

$$\frac{a^2}{\kappa} > 6, \quad (20)$$

for an incident soliton of the form $u(0, t) = a/ch(at)$ injected in one of the two fibers ($a = 1$ above). Secondly let us define the scaling quantities in equations(6)-(7), by using the soliton units, $z = z^{phys}/L_D$, $t = t^{phys}/\tau_0$, $u, v = \sqrt{\frac{2\pi n'_0}{\lambda_0}} L_D B_{1,2}$, and $\kappa = \kappa^{phys} L_D$, where $L_D = \frac{\tau_0^2}{\kappa^{phys}}$ is the dispersion length, and $B_{1,2}$ are the slowly varying amplitudes of the electric field (see appendix). The relation (20) becomes

$$n_2 I_M = n'_2 |B_M^2| > \frac{3}{\pi} \lambda_0 \kappa^{phys}, \quad (21)$$

or, $n_2 I_M > \frac{3}{2} \frac{\lambda_0}{L_c}$ when introducing the switching length $L_c = \frac{\pi}{2\kappa^{phys}}$ defined for the CW linear regime. Using the relation (A3), the low coupling range also writes

$$L_c > 3\pi L_D. \quad (22)$$

IV. SEMICLASSICAL QUANTIZATION OF THE COUPLED FIBER SYSTEM

Before computing the quantum tunneling, let us recall the main differences between the classical solitonic solution and its quantized form. In quantum mechanics a state localized on one side or on the other only is not an eigenstate of the system, because of the possibility of tunneling. Therefore if one starts at ‘time’ zero with all the amplitude on one side (meaning all the probability in one of the two possible asymmetric states), after the time of tunneling this will be transferred to the other side and eventually oscillate between the two sides. It is also possible to inject at the input of the dual core fiber, the quantum ground state, which is symmetrical, with equal amplitude in the two sides. To have a physical image of the process by which the transition occurs between the two asymmetric states, one may recall that the number of photons is not fixed in the quantized soliton, so that it fluctuates in both fibers. Therefore, the fluctuations may bring one fiber into the soliton state, although the other goes to the state without soliton, and the two states switch in the course of time, as studied below.

Let us now outline how to compute the quantum tunneling between the two fibers. Because we have a classical field, the quantization of the coupled equations (6), (7) for the two fibers belongs to the general problem of quantization of field theories. Although this may be done formally, it requires a rather heavy machinery in any case. Fortunately there are various possible short cuts in this derivation. The most obvious one is to reduce PDE's system to a set of ODE's, by using trial functions depending on a certain set of unknown parameters. By refining the choice of trial functions ad infinitum, namely by introducing trial function with more and more parameters, one should converge in principle toward the exact result. But we will merely use the above described trial functions. The Euler–Lagrange condition of stationarity of the action yields a set of dynamical (in ‘time’ z) equation, that can be formally quantized because it has a symplectic structure.

This is what we are going to do, except for one point. It is possible to short cut all this explicit quantization in the WKB limit, where the wave function is expressed by means of the classical Hamilton-Jacobi action, $\Phi = A \exp(i\mathcal{S}/\hbar)$. This is the well-known quasi-classical limit, that restricts oneself to situations where any action involved is typically much bigger than \hbar . This seems a reasonable limit, but it does not necessarily cover all possible situations—we shall come at the end to what seems to be ‘the’ standard experimental situation in this respect. The WKB limit is especially convenient for treating tunneling problems, because it amounts to calculate the imaginary part of the action (which is complex) and to put at the end \hbar at the right place. Indeed the tunneling factor is given by $T = \exp(-2S_E/\hbar)$, at leading order. Here S_E is the imaginary part of the action, which enters then in the modulus of the wave function as a real exponent (instead of the usual imaginary exponent relevant for the classical limit of quantum mechanics). This imaginary part of the action is calculated by two steps. First one has to change the conjugate variables (q, p) into (q, ip) in the classical Hamilton-Jacobi formulation of quantum mechanics, the Hamiltonian $H(q, p)$ becoming $H(q, ip)$. Secondly one is left with a problem of extremalization of a new action, the Euclidean action, that is formally another problem of classical mechanics. For instance in the often presented problem of a particle of energy E in a double well potential $V(q)$, with Hamiltonian $H = \frac{p^2}{2m} + V(q)$, the Euclidean action is calculated with the abbreviated action [13]

$$\mathcal{S}_E = \int_{q[0]}^{q[z_f]} p dq \quad (23)$$

derived from the Hamiltonian $H_E = -\frac{p^2}{2m} + V(q)$ but with the same energy as the one of the classical motion. For the potential this means that it gets rotated by 180 degrees, thus exhibiting two “hills” of maximal energy. The values of $q[0]$ and $q[z_f]$ in equation (23) are those of the classical turning points defined by $E = V(q)$. To calculate \mathcal{S}_E , one has to find a trajectory joining these points, namely to calculate an Euclidean path integral. This is performed by solving the Hamilton equations for the Euclidean Hamiltonian

$$\begin{cases} \frac{\partial q}{\partial z} = \frac{\partial H_E}{\partial p} \\ \frac{\partial p}{\partial z} = -\frac{\partial H_E}{\partial q} \end{cases} \quad (24)$$

by taking as initial conditions, the known value $q[0]$, and an unknown value $p[0]$. By varying the latter value, one finally converges towards a trajectory ending at $q[z_f]$, which provides the action defined in equation (23). Note that equations (24) are obtained from the classical Hamiltonian system (which is identical to equation (24) but with H in place of H_E), by changing z in iz , and p in ip . The change to an imaginary “time” (from z to iz here) amounts to go from a Minkowskian to an Euclidean metric. Therefore equations (24) are called “Euclidean equations of motion”, and their classical solution joining the two “vacua” of the double-well potential, often named “kink solution”, is an example of an instanton [14] in quantum mechanics.

In the above example the variables (p, q) are the impulsion and position of a particle in a 1D potential. Generalization to cases of a multidimensional set of generalized coordinates and momenta leads to similar relations [13].

A. Semi-classical Action.

To put all those principles in practice we have to formalize the dynamical system (equation (15)) in terms of canonically conjugate variables. Once this is done, the Euclidean equations of motion are found by multiplying the “time” z and the momenta by i . As noted in section III, the reduced equations (15) are those of an Hamiltonian system with two degrees of freedom, therefore a simple choice for conjugate variables (q_j, p_j) with $j = 1, 2$, is to take

the pair $(x = \cos(2\Theta), y = \frac{\pi^2}{6\eta^2})$ as coordinates and $(2\Psi, q)$ as their conjugate momenta. The Euclidean Hamiltonian is obtained from the classical one in equation (19), by changing $\cos(2\Psi)$ into $\cosh(2\Psi)$, and q^2 into $-q^2$. It becomes

$$\mathcal{H}_{E,\text{trial}} = -2\kappa \cosh(2\Psi) \sin(2\Theta) + \frac{1}{3}\eta \sin^2(2\Theta) - \frac{2\eta}{3} + \frac{\eta^2}{3} - \frac{\pi^2 q^2}{3\eta^2}. \quad (25)$$

The semi-classical dynamics is then driven by the new set of four (Euclidean) equations, that are the Hamilton equations for the conjugate variables (q_j, p_j) , deduced from the Euclidean Hamiltonian (25)

$$\begin{cases} \frac{dx}{dz} = -2\kappa \sinh(2\Psi) \sqrt{1-x^2} \\ \frac{d(2\Psi)}{dz} = \frac{2}{3}\eta x - 2\kappa \cosh(2\Psi) \frac{x}{\sqrt{1-x^2}} \\ \frac{d\eta}{dz} = 2q\eta \\ \frac{dq}{dz} = 2q^2 + \frac{2}{\pi^2}(\eta^4 - \eta^3 \frac{1+x^2}{2}) \end{cases} \quad (26)$$

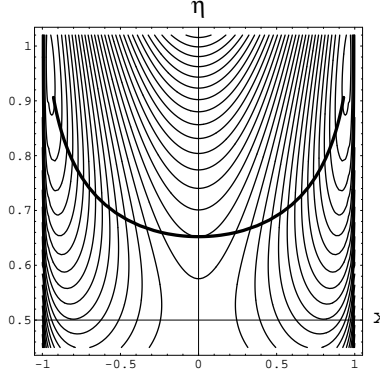


FIG. 3: Quantum trajectory superposed to the potential for $\kappa = 0.1$: only two asymmetric solutions are stable.

As in the case of a particle in a double well potential, calculating the probability for the soliton to tunnel through a classically forbidden region (H_{pot}) with the Minskowskian space path integral, corresponds to calculating the transition probability to tunnel through a classically allowed region ($-H_{\text{pot}}$) in the Euclidean path integral, with the action

$$\mathcal{S}_E = \int_{q_L}^{q_R} (p_1 dq_1 + p_2 dq_2) \quad (27)$$

where q_L, q_R are the coordinates of the turning points. To perform the integration giving the action, it is enough to choose a convenient integration path in the Euclidean plane connecting the two minima $M(x_M, \eta_M)$ of the classical potential in Fig. 3, which become the maxima of the Euclidean potential. For small values of κ , one has $\mathcal{H}_{\text{pot},M} \approx -\frac{1}{3}(1 + 18\kappa^2)$. In the present case it is easier to carry the integral from $x_0 = 0$ up to x_M . We set the value of the Hamiltonian \mathcal{H}_0 close to $\mathcal{H}_{\text{pot},M}$. Because of the symmetry of the heteroclinic trajectory joining the two extrema of the potential, we have to choose the initial condition $q_0 = 0$. Then the initial value of the phase difference Ψ_0 is deduced from equation (25), and we have only one initial parameter to adjust, η_0 , in order that the trajectory ends with a vanishing impulse $q_f = \Psi_f = 0$ close to the extrema $M(x_M, \eta_M)$ in the plane (x, η) . The integration path is shown in Fig.3. The action along the semi-classical trajectory is given by the expression (27) that writes with our notations

$$\mathcal{S}_E = 2 \left| \int_{z=0}^{z=z_f} \left[-4\Psi(z)\kappa \sinh(2\Psi(z)) \sqrt{1-x(z)^2} - \frac{2\pi^2}{3} \frac{q^2(z)}{\eta^2(z)} \right] dz \right|. \quad (28)$$

The numerical result of the integration is shown in Fig. 4 which displays the action as a function of the coupling parameter κ in a logarithmic scale. In the domain of existence of the asymmetric solution, $\kappa^{-1} > 6$ in Fig. 1, the action clearly displays a logarithmic dependence with respect to the coupling, we have the law

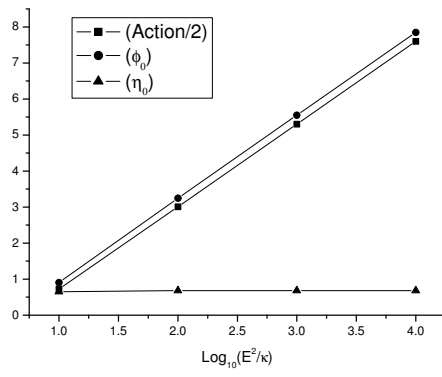


FIG. 4: Action/2 (squares), Maximum of the impulse phase Φ_0 (circles), and η_0 (triangles) versus the scaled coupling parameter

$$\mathcal{S}_E = 2 \ln(\kappa_c/\kappa) \quad (29)$$

that holds true with a precision better than 1 per cent over many decades, with the numerical value $\kappa_c = 0.2$ slightly higher than the bifurcation one 0.18. Note that the relation (29) holds true except in the close vicinity of the bifurcation point, not visible in Fig. 4. The \ln -dependence in equation (29) follows straightforwardly from the substitution of exponentials for the hyperbolic sine in the equation of motion for Euclidean dynamics. We also report in Fig. 4 the dependence of Ψ_0 and η_0 as function of κ . At $x = 0$ the solution becomes transiently symmetric, $\sin(2\Theta) = 0$, but its width is different from the symmetric value, $\eta_0 \neq \frac{1}{2}$, and the impulse Ψ_0 is maximum. We show that Ψ_0 evolves very much as the action, while η_0 is quite constant. Actually the heteroclinic trajectory drawn in Fig.(3) passes through the abscissa $x = 0$ approximately at the ordinate $\eta \sim 0.67$ whatever the value of the coupling constant, while the impulse here increases like $\ln(\kappa_c/\kappa)$. This result shows the leading role of the conjugate variables Ψ and $x = \cos(2\Theta)$ in the dynamics. At this stage it is interesting to compare the latter result (29) with the action derived by a simpler choice of trial functions, based on the hypothesis of constant width soliton (and of no chirp), as proposed by Paré [8] and Kivshar [9]. In these simpler cases, one obtains a single degree of freedom Hamiltonian dynamics. The approximate calculation of the Euclidean action may be done analytically, and leads to similar results in both cases. With the notations of Kivshar, for example, using as conjugate variables (Φ, Δ) , the calculation of the action amounts to carry the integral $\mathcal{S}_E = \int_{-1}^{+1} \Phi(\Delta) d\Delta$, the function $\Phi(\Delta)$ being given explicitly in [9]. In the limit of a small coupling and with $\kappa = \gamma^{-1}$, the equation for Φ reduces, at leading order, to $\Phi \approx i \ln(\gamma) + \hat{\Phi}$ where $\hat{\Phi}$ is the solution of $\frac{1}{I(\Delta)} = e^{i\hat{\Phi}}$ that is of order 1. Therefore in this limit γ large (equivalent to small coupling), $\Phi \approx i \ln(\gamma)$ so that the action associated to tunneling is just $S \approx 2i \ln(\frac{\gamma}{\gamma_c})$, where γ_c is a constant.

Summarizing the Euclidean action obeys the law (29) in all cases of trial functions we have considered, i.e. for a single degree of freedom Hamiltonian as well as with two degrees of freedom. Consequently, it does not seem necessary to refine more our model to obtain the information we need, i.e. the order of magnitude of the tunneling amplitude.

B. Tunneling factor.

The possibility for the soliton to tunnel from one fiber to the other in real space, is measured by the transmission coefficient, with the expression $T = |\frac{F}{A}|^2$ in a double well tunneling problem, with F, A the amplitudes of the transmitted and incident waves, respectively [5]. It has already been noted that the transmission is given by $T = \exp(-2S/\hbar)$ at leading order. In practice S is the "physical action", having the same dimension as \hbar . Therefore, to calculate the "true" transmission for the soliton in the two coupled fibers one has to multiply the dimensionless action S_E by an appropriate coefficient $s^{(1)}$ depending on the properties of the fiber and of the characteristics of the EM wave, this giving lastly the "physical action" $S_E^{phys} = s^{(1)} S_E$ which has the dimension of \hbar . As shown in the appendix

$$2s^{(1)}/\hbar = \gamma/(\omega_0\tau_0)^3 \quad (30)$$

where

$$\gamma \sim 8\sigma\epsilon_0 c^3 k''^2 / (n'_2 \hbar). \quad (31)$$

The value of γ depends on the fiber parameters n'_2 and σ , cross section of the fiber. Let us consider $10\mu m^2$ area silica fibers with $n'_2 = 2.610^{-22}(m/V)^2$ as given in ([16]), ([15]). With the values of coefficients given in the appendix in MKS units, the coefficient γ is about 3.610^8 .

With equations (29)-(31), the transmission coefficient

$$T = \exp - \left[\frac{2\gamma}{(\omega_0 \tau_0)^3} \ln(\kappa_c / \kappa) \right], \quad (32)$$

or

$$T = \left(\frac{\kappa}{\kappa_c} \right)^{\frac{2\gamma}{(\omega_0 \tau_0)^3}}, \quad (33)$$

behaves as a power law, that is smoother than the usual exponential in tunneling amplitudes. The tunneling is possible when the exponent in equation (32) is "not too big". In the semi-classical regime considered above, the phase of the wave-function is derived by expansion at lowest order with respect to \hbar . This requires that the exponent $\ln(T) = \frac{2\gamma}{(\omega_0 \tau_0)^3} \ln(\kappa_c / \kappa)$ is much larger than unity, then the probability of tunneling is obviously weak. When the exponent becomes smaller or of order unity, one is in the "pure quantum limit", and the previous derivation is no more valid, since the wave-function cannot reduce to its first order term in \hbar . Nevertheless we can assert by continuity argument that tunneling continue to exist, and that it is likely much more efficient. The boundary between these two limits can be defined by

$$\frac{\gamma}{(\omega_0 \tau_0)^3} \ln(\kappa_c / \kappa) = 1. \quad (34)$$

This dependence is drawn in Fig. 5 for the value of γ given above. The quantum regime is reached as soon as the pulse duration is longer than a ps . Therefore quantum tunneling seems within reach of present days experiments.

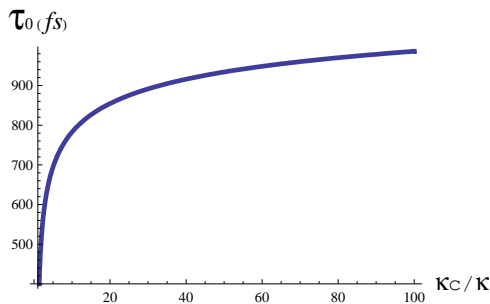


FIG. 5: Boundary between the quantum and semi-classical regime, for a silica core fiber. The quantum regime stays above the frontier.

C. Quantum switching.

To estimate the typical length needed for the soliton to tunnel from fiber to fiber, we reason as follows. We estimate first the time scale for the quantum tunneling. We split the wavefunction into the 'right' amplitude, Φ_R , and the left one, Φ_L , each one being for the state in one fiber only. Because of the tunneling those states are not eigenstates but split into two eigenstates, one even (the ground state) Φ_S and the other odd, Φ_A , under permutation of the two fibers. One has

$$\Phi_L = (\Phi_S - \Phi_A)/\sqrt{2}, \text{ and } \Phi_R = (\Phi_S + \Phi_A)/\sqrt{2}.$$

The energy difference between the symmetric and antisymmetric state gives, via the Planck-Einstein relation, the typical tunneling time. Let A be half of this energy difference. If at time zero the soliton is on the right fiber, the evolution of its amplitude later on is given by

$$\Phi(t) = \frac{1}{\sqrt{2}}(\Phi_S e^{-iE_S t/\hbar} + \Phi_A e^{-iE_A t/\hbar}). \quad (35)$$

Therefore the amplitude in one fiber oscillates with the period

$$T_{osc} \sim h/2A. \quad (36)$$

In the following derivation, we approximate the energy splitting in each well by using the standard result for a particle of momentum $p(x)$ in a double-well:

$$2A = \frac{\hbar\omega}{\pi} \exp\left(-\frac{1}{\hbar} \int_{-a}^a |p| dx\right) \quad (37)$$

where $[-a, a]$ is the x -range under the barrier for the given energy, and ω the pulsation of the wave-function in the bottom of the well. For a quadratic potential $V(x)$, of curvature V'' around the minimum x_M , the pulsation of a particle of mass m is such that

$$V'' = m\omega^2 \quad (38)$$

The mass of the particle is deduced from its momentum under the barrier of height U_0 , at $x = 0$, where $p_0^2/2m = U_0$. Therefore the pulsation writes

$$\omega = \sqrt{\frac{2V''U_0}{p_0^2}} \quad (39)$$

where all quantities are in physical units. Note that the dimensions are $[P][x] = [S]$ and $[V''] = [V]/[x^2]$, therefore the dimension of x plays no role. For the fiber problem, we shall consider only one set of conjugate variables, (x, Ψ) , neglecting the η dependance of the potential, which plays a secondary role, moreover the physical quantities in equation (39) have to be expressed in terms of the reduced ones, and the "time" period of equation (36) becomes a spatial period. This writes

$$\begin{cases} p_0 = S^{(1)}\Psi_0 \\ V'' = W^{(1)}v'' = \omega_0 S^{(1)}v'' \\ U_0 = W^{(1)}\Delta V \\ T_{osc} = Zk_0''/\tau_0 \end{cases},$$

where the curvature V'' and the height ΔV of the potential barrier are deduced from equation (19), that gives

$\Delta V = 0.037 + 6\kappa^2$, and $v'' = \frac{2}{3^3}\kappa^{-2}$. Moreover the numerical results in Fig. 4 give $\Psi_0 = \ln(\kappa_c/\kappa)$. In the above relations Z is the "true" spatial period along the optical axis of the fibers, obtained from equation (A1) after dividing all terms by k_0''/τ_0 to obtain a soliton of half-mass equal to unity as assumed in the present section.

With these expressions, the equation (36) becomes

$$\omega_0 T_{osc} = Z \frac{\omega_0 k_0''}{\tau_0} = \frac{2\pi^2 \Psi_0}{\sqrt{2v''\Delta V}} \exp S/\hbar, \quad (40)$$

with $S/\hbar = \frac{\gamma}{(\omega_0 \tau_0)^3} \ln(\kappa_c/\kappa)$.

Since the probability of finding the soliton in a given fiber oscillates with respect to the spatial variable z with a wavelength Z , it also oscillates in time from one fiber to the other with the period $\tau = nZ/c$, at a given z . Using the numerical values given in the appendix for standard fibers, the period of the switching depends on the two parameters κ and τ_0 . The frequency $\nu = c/nZ$, and the spatial period Z are drawn in fig.(6), as function of the coupling parameter ratio κ_c/κ . We have chosen two values of pulse duration, $\tau_0 = 0.6ps$ (dashed line, corresponding to $r = 1$), and $1.3ps$ (solid line, $r = 0.1$), which are respectively below, and above the frontier drawn in Fig.(5). More precisely

the dashed line stands into the "semiclassical" regime as soon as κ_c/κ is larger than few units where the WKB approximation is valid, whereas the solid line corresponds to the "purely quantum" regime. The two lines displays high frequencies, ranging from hundred of Mhz , towards tens of Ghz , that could be interesting for applications to high speed transmission. Note that while the solid line corresponds to the pure quantum regime, where the WKB approximation used here is not valid, we infer that it could be possible that going beyond the WKB approximation, would lead to even higher frequencies. It could then lead to shorter switching lengths than those displayed in Fig. (6-b)). In the semi-classical regime, the switching length Z is longer, nevertheless it is much shorter than the half period of switching in the CW linear case, $L_c = \frac{\pi}{2\kappa_{phys}}$. Indeed a pulse duration $\tau_0 = 0.6ps$, and a silica fiber, one has $L_D = 16m$, that gives $L_c = 125 \frac{\kappa_c}{\kappa}$ when using $k_c = 0.2$ (see Fig.(4)). For $\frac{\kappa_c}{\kappa} = 100$, the linear half period is $L_c = 12.5km$, which is several order of magnitudes longer than the semiclassical switching length $Z = 3m$ (dashed curve).

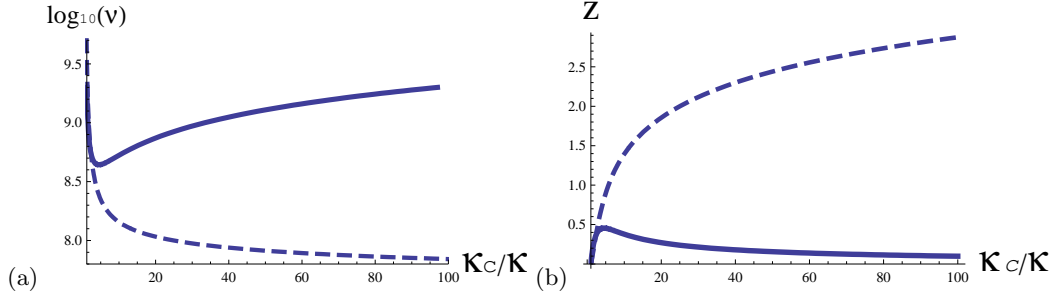


FIG. 6: (a) Frequency of the periodic switching $\nu = 1/\tau$, in Log scale, with ν in Hz ; (b) Spatial period Z , in m , as function of the ratio κ_c/κ , for the case $\tau_0 = 0.6ps$ ($r=1$, dashed line) and $1.3ps$ ($r=0.1$, solid line). For $\frac{\kappa_c}{\kappa} = 100$, the period is $Z = 10cm$ for the solid line, and about $3m$ for the dashed line .

V. SUMMARY AND DISCUSSION

Even though the tunneling phenomenon is very familiar in many wave-propagation problems, where the "true" wave-vector \vec{k} becomes $i\vec{k}$ after passing under a classical barrier, (as in the case of evanescent waves in the Fresnel theory), it appears in the present context in a slightly unusual form: starting from the classical model (6), (7) for the field envelope, which looks strangely similar to the Schrödinger equation, our treatment based on the approximate trial functions leads finally to the Euclidean system (26) which is not the Schrödinger equation for a wave function.

Within the trial functions approximation, the WKB or quasiclassical limit gave us the possibility of estimating rather easily the rate of *quantum* tunneling of a single soliton from one fiber to the other, even though it should remain *classically* in the same fiber forever. We found that this rate of tunneling is not small and could well be within reach of present day-experiments.

In the frame of the WKB approximation, we are trying to extend our results by getting rid of the trial functions approximation. Our aim is to check if the relation (29), that has been shown here to survive when going from two to four unknown parameters in the trial functions approximation, is valid beyond this approximation. The calculation is heavier than the one presented here, because the time t is now considered as an infinite dimensional parameter, then the semi-classical trajectory must be calculated from a set of 4 coupled PDE's, in place of the 4 coupled ODE's (26) solved here. To derive these PDE's, we can choose for example $(\Im(u), \Im(v))$ and $(\Re(u), \Re(v))$ as set of conjugate variables (\mathbf{p}, \mathbf{q}) for the classical system (6), (7), with Hamiltonian

$$\mathcal{H}_{NLS} = \int dt \left[\frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 - \frac{1}{2} |u|^4 + \frac{1}{2} \left| \frac{\partial v}{\partial t} \right|^2 - \frac{1}{2} |v|^4 - \kappa (u\bar{v} + v\bar{u}) \right]. \quad (41)$$

The Euclidean version of equations (6), (7) is then obtained by changing $(z, \Im(u), \Im(v))$ into $(iz, i\Im(u), i\Im(v))$. The important point is that the tunneling factor does not depend on the choice of (\mathbf{p}, \mathbf{q}) , while the Euclidean system obviously does. Finally, the heteroclinic Euclidean trajectory is the solution connecting two of the classically permitted orbits, from $z = -\infty$ to $z = +\infty$. For a given energy E , these are defined by the integro-differential equation $\mathcal{H}_{NLS} = E$, where $\Im(u) = \Im(v) = 0$, and correspond to the two asymmetrical solitons.

Because we have found that the Euclidean action S_E^{Phys} can be of order of \hbar or even smaller in realistic experimental conditions, it could even happen that the WKB quasiclassical approximation is not valid anymore for computing the

rate of transfer from one fiber to the other. Usually the order of magnitude of the action involved in the soliton picture, even in a single fiber, is tacitly assumed to be far bigger than \hbar , which is an assumption distinct from the one of a soliton made of many photons. Indeed the soliton picture addresses perturbations to this ‘bound state’ of many photons that may be small enough to imply variations of the action of order of \hbar , and so require some sort of (‘second’) quantization. We plan to come to this general question in future work, and outline here some of the estimated problems.

A treatment using the trial function, but valid beyond the WKB approximation, is obviously more complicated than what we did here, and perhaps questionable. Indeed it needs to consider both the trial functions and their parameters as operators. Moreover it amounts to assume that the fluctuations in t and z are decoupled, and, last but not least, our result derived in the WKB approximation likely signals that the assumption behind the classical (meaning non quantum) theory for describing soliton in coupled fibers does not hold anymore and that the quantum picture has to be used from the start, which makes it theoretically challenging.

We assumed that every phenomenon under study involved solitons seen as a coherent quantum objects. We argued that this requires that any typical time, the tunneling time in particular, is far shorter than the coherence time. This coherence time is of order of t_c/N , with t_c coherence time of a single photon in the soliton, i.e. its mean-free flight time without change in phase or frequency. Because of the division by N this may be a very short time. At times longer than the coherence time any physical effect related to the quantum coherence between states of solitons propagating in either fiber is washed out. The final state, as described in the density matrix formalism, is a state of equal probability of the soliton on either side without nondiagonal element. The experimental manifestation of this state will be a probability 1/2 of observing a soliton in either fiber without any possible interference between the states on either side. Somehow this will bring the system back to a fully classical state, except that this classical state has a probabilistic underpinning that is absent from the classical system: in the fully classical system the soliton remains always in the same fiber, although in the quantum one its final state has a probabilistic nature.

Looking at the other side of the coin one realizes that, because the soliton is a composite object, and if it remains coherent during a sufficiently long amount of time, its phase is the phase of a single photon multiplied by the number of photons. Therefore any interference experiment between coherent soliton states will have much narrower interfringe than with a single photon or incoherent photons, this interfringe being the one for a single photon divided by the number of photons making the coherent soliton. This could be of interest for gyroscopes based on the Sagnac effect [17].

Indeed a central issue concerning the observability of the tunneling effect we present in this communication is the one of the quantum coherence of the soliton, related itself to all dissipative effects that can break up this coherence, and that makes the main topic discussed in the present special issue. Nevertheless, even if the coherence is limited, there is still tunneling, but at a reduced rate [3]. In that case we suggest to use twin fibers with coupling coefficient κ periodically modulated in z , in order to stimulate the switching process.

Acknowledgments

Elisabeth Ressayre et Jean Ginibre are gratefully acknowledged for stimulating discussions, and Laurent Di Menza for providing us a code for simulations of the NLS model with transparent boundary conditions.

APPENDIX A: PHYSICAL UNITS

This appendix is about the relationship between quantities measured in physical units for a standard fiber carrying solitons and the dimensionless quantities used in the bulk of our paper. We relate first the number of photons in a typical soliton to its time duration, a duration called τ_0 that we shall use afterwards to give various order of magnitudes pertinent to our problem.

We use the standard expression of the electric field in a fiber, written with the same notations as in the book by Newell and Moloney [15]. The electric field of the EM wave in the fiber is modeled by the wave-packet expression, $E = R(x_t)B(z, t) \exp(i\omega_0 t - k_0 z) + c.c.$ which obeys in a first approximation Maxwell’s equations, when the duration of the pulse is not too short, x_t being the transverse coordinate, and $R(x_t)$ the dimensionless radial amplitude with $\int |R(x')|^2 dx' = \sigma$ as the core area.

Taking $\frac{\delta\omega}{\omega_0}$ as a small parameter, $\delta\omega$ being the frequency width of the pulse, one obtains the NLS equation written in variables $z, \tau = t - z/v_g$, the nonlinear and the dispersion term having opposite signs:

$$i \frac{\partial B}{\partial z} + \frac{k_0''}{2} \frac{\partial^2 B}{\partial \tau^2} + \frac{2\pi n_2'}{\lambda_0} |B|^2 B = 0. \quad (A1)$$

Note that we turned to the standard writing of the coefficients of the NLS equation, n'_2 being the modulus of the coefficient of the cubic Kerr effect and k_0'' the modulus of the second derivative of the wavenumber k_0 with respect to the frequency of the EM wave.

The soliton solution is

$$B(z, t - z/v_g) = B_m \text{sech}((t - z/v_g)/\tau_0), \quad (\text{A2})$$

with

$$\beta |B_m|^2 = 1/\tau_0^2. \quad (\text{A3})$$

where $\beta = \frac{2\pi n'_2}{\lambda_0 k_0''}$. Its energy is

$$W = \int P(t) dt = \sigma \int I(t) dt, \quad (\text{A4})$$

where $I(t)$ is the optical intensity measured in watt per square meter, and $P(t)$ is the Poynting vector integrated across the fiber section, with a result expressed in Watts:

$$P(t) = \int (\mathbf{E} \wedge \mathbf{H}) \cdot \mathbf{z} dS \quad (\text{A5})$$

where \mathbf{z} is the unit vector in the direction of propagation.

The magnetic field in the wave (supposing it is linearly polarized with the electric field in the x -direction) is $H_x = n c \varepsilon_0 E_y$, n index of refraction. For a material with instantaneous response, after averaging over one period of the field oscillations, one finds

$$\overline{P}(t) = 2c\varepsilon_0 \sigma n |B(t)|^2. \quad (\text{A6})$$

At leading order, i.e. by taking into account the linear part of refractive index, $n = n_0$, this yields

$$I^{(0)}(t) = 2c\varepsilon_0 n_0 |B(t)|^2. \quad (\text{A7})$$

Whence the energy of the pulse is:

$$W^{(0)} = N\hbar\omega = 2n_0 c \varepsilon_0 \sigma \int |B(t - z/v_g)|^2 dt = 4n_0 c \varepsilon_0 \sigma \frac{\lambda_0 k_0''}{2\pi n'_2} \frac{1}{\tau_0}. \quad (\text{A8})$$

Finally the relationship between the photon number and the pulse duration writes

$$N\tau_0 = 4n_0 \varepsilon_0 \sigma \left(\frac{\lambda_0}{2\pi}\right)^2 \frac{k_0''}{n'_2 \hbar}. \quad (\text{A9})$$

Similarly the action is

$$S^{(0)} = N\hbar = W^{(0)}/\omega, \quad (\text{A10})$$

at leading order.

In MKSA units, with standard values (see [15]) of optical fibers composed of silica cores, this gives:

$$n_0 = 1.5$$

$$\epsilon_0 = 0.89 \cdot 10^{-11} F/m, \text{ or } \epsilon_0 c = 1/Z_0 \text{ with } Z_0 = 377\Omega \text{ the impedance of free space,}$$

$$\hbar = 10^{-34} J.s$$

$$\lambda_0 = 1.55 \cdot 10^{-6} m$$

$$\sigma \sim 10^{-11} \text{ for a } 10\mu m^2\text{-area fiber.}$$

$$k_0'' = 2.2 \cdot 10^{-26} s^2/m$$

$$n'_2 = 2.6 \cdot 10^{-22} (m/V)^2$$

With these data, the number of photons in the pulse of duration τ_0 , measured in seconds obeys the relation:

$$N\tau_0 \sim 3.10^{-5}, \quad (\text{A11})$$

that gives $N \sim 3.10^7$ photons for a ps -pulse.

Let us note that the nonlinear index of refraction n'_2 may be several orders of magnitude larger, when using other materials. For example, in the experiment of Wa et al. [18], the optical switch was studied in multiple quantum well wave-guides, with $n'_2 = 10^{-13}(m/V)^2$.

Coherent part of the energy and action

Let us write the energy and action as

$$\mathcal{W} = W^{(0)} + W^{(1)}, \quad (\text{A12})$$

and

$$\mathcal{S} = S^{(0)} + S^{(1)}. \quad (\text{A13})$$

The dominant contributions are proportional to the linear part of the refractive index, namely a term contained in the Maxwell equation. The subdominant contributions $W^{(1)}$ and $S^{(1)}$, correspond to the terms contained in the envelope equation, they are perturbations to the dominant effects calculated above. For two coupled fibers, these perturbations result from balanced effects of dispersion, coupling and nonlinearity. They are proportional to the energy and action of the the dimensionless NLS equation (6-7),

$$\begin{cases} W^{(1)} = w^{(1)} \mathcal{H}_{\text{NLS}} \\ S^{(1)} = \frac{w^{(1)}}{\omega_0} \mathcal{S}_{\text{NLS}} \end{cases} \quad (\text{A14})$$

where the scaled energy \mathcal{H}_{NLS} is defined in equation (41), and the action in equation (10).

The coefficient $w^{(1)}$ may be calculated by using the expression of the Poynting vector (A6) valid for dispersionless Kerr media, where

$$n = n_0 + n_2 I = n_0 + n'_2 |B(t)|^2, \quad (\text{A15})$$

This gives $W^{(1, Kerr)} = 2n'_2 c \epsilon_0 \sigma \int dt |B(t)|^4$.

Taking the hyperbolic secant solution (A2-A3), one obtains $W^{(1, Kerr)} = 2n'_2 c \epsilon_0 \sigma |B_M(t)|^4 \tau_0 \int dt \text{sech}^4(t)$, or

$$W^{(1, Kerr)} = -4n'_2 c \epsilon_0 \sigma \frac{1}{\beta^2 \tau_0^3} \mathcal{H}_{\text{NLS}}^{\text{Kerr}}, \quad (\text{A16})$$

where $\mathcal{H}_{\text{NLS}}^{\text{Kerr}}$ is the Kerr contribution of the Hamiltonian (second term in the r.h.s. of equation 5). This correction is the "coherent" part of the energy in the sense that it is proportional to the square of the intensity, or of the photon number. Finally we are ready to express the physical value of the Hamiltonian and action associated to a soliton whose temporal width is scaled to τ_0 as it was assumed in sections 3-4, by using the expression (A14) with

$$w^{(1)} \simeq 4n'_2 c \epsilon_0 \sigma \frac{1}{\beta^2 \tau_0^3}. \quad (\text{A17})$$

This allows in particular to express concretely the constraint that nonlinear effects are small, that is that $W^{(1)} \ll W^{(0)}$, a condition equivalent to

$$\frac{W^{(1)}}{W^{(0)}} = \frac{2}{3} \frac{n'_2}{n_0} \frac{1}{\beta \tau_0^2} \ll 1. \quad (\text{A18})$$

Finally the physical action associated to the quasi-classical trajectory is approximately given by the expression

$$\mathcal{S}_E^{phys} = \frac{w^{(1)}}{\omega_0} S_E. \quad (\text{A19})$$

The quantum tunneling coefficient $T = \exp(-2S_E^{phys}/\hbar)$ is expected to be experimentally observable when the physical action is not too large with respect to \hbar . When S_E^{phys} becomes of order \hbar , we can infer that we are in the "pure quantum" regime. The frontier between these two regimes may be drawn in the space parameters (κ, τ_0) by introducing the parameter

$$\gamma = \frac{8c^3 \epsilon_0 \sigma k''_0^2}{\hbar n'_2} \quad (\text{A20})$$

that allows to write

$$2 \frac{w^{(1)}}{\hbar \omega_0} = \frac{\gamma}{(\omega_0 \tau_0)^3}. \quad (\text{A21})$$

With the data given above for standard fibers, $\gamma = 3.610^8$, the semi-classical regime stands below the curve drawn in section IV (see Fig. (5)). Consequently pulses longer than one *ps* typically stands in the pure quantum regime. In conclusion we predict quantum tunneling for realistic conditions of soliton propagation in two coupled fibers.

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